

Estimates of complex eigenvalues and a inverse spectral problem for the transmission eigenvalue problem

Xiao-Chuan Xu¹, Chuan-Fu Yang², Sergey A. Buterin³
and Vjacheslav A. Yurko⁴

Abstract. This work deals with the interior transmission eigenvalue problem: $y'' + k^2 \eta(r) y = 0$ with boundary conditions $y(0) = 0 = y'(1) \frac{\sin k}{k} - y(1) \cos k$, where the function $\eta(r)$ is positive. We obtain the asymptotic distribution of non-real transmission eigenvalues under the suitable assumption for the square of the index of refraction $\eta(r)$. Moreover, we provide a uniqueness theorem for the case $\int_0^1 \sqrt{\eta(r)} dr > 1$, by using all transmission eigenvalues (including their multiplicities) along with a partial information of $\eta(r)$ on the subinterval.

Keywords: Transmission eigenvalue problem, Scattering theory, Complex eigenvalue, Inverse spectral problem

2010 Mathematics Subject Classification: 35P25; 34L15; 34A55

1. INTRODUCTION AND MAIN RESULTS

Consider the interior transmission problem

$$y'' + k^2 \eta(r) y = 0, \quad 0 < r < 1, \quad y(0) = 0 = y'(1) \frac{\sin k}{k} - y(1) \cos k, \quad (1)$$

where the square of the index of refraction $\eta(r)$ is a positive function in $W_1^2[0, 1]$ with the natural assumption $\eta(1) = 1$ and $\eta'(1) = 0$. The k^2 -values for which the problem (1) has a nontrivial solution $y(r)$ are called *transmission eigenvalues*. The problem (1) appears in the inverse scattering theory for a spherically stratified medium, which consists in determining the function $\eta(r)$ from transmission eigenvalues. To study the inverse spectral problem, one has to investigate the property of transmission eigenvalues, such as, the existence of real or non-real eigenvalues and their asymptotic distribution.

We introduce two key quantities. Denote

$$a := \int_0^1 \sqrt{\eta(r)} dr, \quad (2)$$

¹Department of Applied Mathematics, School of Science, Nanjing University of Science and Technology, Nanjing, 210094, Jiangsu, People's Republic of China, *Email:* xiaochuanxu@126.com

²Department of Applied Mathematics, School of Science, Nanjing University of Science and Technology, Nanjing, 210094, Jiangsu, People's Republic of China, *Email:* chuanfuyang@njut.edu.cn

³Department of Mathematics, Saratov University, Astrakhanskaya 83, Saratov 410012, Russia, *Email:* buterinsa@info.sgu.ru

⁴Department of Mathematics, Saratov University, Astrakhanskaya 83, Saratov 410012, Russia, *Email:* yurkova@info.sgu.ru

which is explained physically as the time needed for the wave to travel from $r = 0$ to $r = 1$. Introduce the *characteristic function*

$$d(k) := y'(1, k) \frac{\sin k}{k} - y(1, k) \cos k, \quad (3)$$

where $y(r, k)$ is the solution of $y'' + k^2 \eta(r) y = 0$ with the initial conditions $y(0, k) = 0$ and $y'(0, k) = 1$. Obviously, the transmission eigenvalues coincide with the squares of zeros of $d(k)$.

The problem (1) was first studied by McLaughlin and Polyakov [11], they showed that if $a \neq 1$ then there are infinitely many real eigenvalues $\{(k'_n)^2\}_{n \geq n_0}$, which have the asymptotics

$$(k'_n)^2 = \frac{n^2 \pi^2}{(a-1)^2} + \frac{1}{a-1} \int_0^a q(x) dx + o(1), \quad n \rightarrow \infty, \quad (4)$$

where $q(x)$ is defined in (9). Colton and co-authors [6, 7] studied the existence of the non-real transmission eigenvalues. They showed that if $a \neq 1$ and $\eta(r)$ is non-constant near $r = 1$ then there exists infinitely many real and non-real transmission eigenvalues. Some aspects of the asymptotics of large (real and non-real) transmission eigenvalues for the case $a = 1$ were discussed in [16].

For the inverse spectral problem, many scholars contribute a lot of works (see [1–5, 11, 15, 18] and the references therein). However, for the case $a > 1$ there are only a few results. It is known [5, 11] that the determination of $\eta(r)$ on $[0, 1]$ with $\eta(1) = 1$ and $\eta'(1) = 0$ is equivalent to the determination of $q(x)$ on $[0, a]$ defined in (9). McLaughlin and Polyakov [11] first showed that if $a > 1$ and $\eta(r)$ is known a priori on a subinterval $[\varepsilon_1, 1]$ with ε_1 satisfying

$$\int_{\varepsilon_1}^1 \sqrt{\eta(r)} dr = \frac{a+1}{2}, \quad (5)$$

then $\eta(r)$ on $[0, \varepsilon_1]$ is uniquely determined by the transmission eigenvalues $\{(k'_n)^2\}_{n \geq 1}$ satisfying (4), where $\{(k'_n)^2\}_{n=1}^{n_0-1}$ may be non-real. In 2013, Wei and Xu [15] suggested to specify all transmission eigenvalues (including their multiplicities) and the norming constants, corresponding to the real eigenvalues, to obtain the unique determination of $\eta(r)$ on $[0, 1]$.

In this paper, we shall give a further discussion for the distribution of non-real transmission eigenvalues and the uniqueness theorem of the inverse spectral problem. We will give not only the existence of non-real eigenvalues but also their asymptotic behavior. Moreover, we prove a uniqueness theorem for the inverse spectral problem in the case $a > 1$, by using the known information on $\eta(r)$ on the subinterval and all eigenvalues (including real and non-real).

The main results in this article are as follows.

Theorem 1.1. *Assume that $\eta \in W_1^{m+3}[0, 1]$ for some nonnegative integer m . If $\eta(1) = 1$, $\eta^{(u)}(1) = 0$ for $u = \overline{1, m+1}$ and $\eta^{(m+2)}(1) \neq 0$, then the non-real*

zeros of $d(k)$ in the right half-plane consist of two sequences $\{k_n^\pm\}$ having the following asymptotic behavior, when $n \rightarrow \infty$,

(i) $a \neq 1$

$$k_n^\pm = n\pi \pm \frac{i}{2} \log \left(\frac{4(2n\pi i)^{m+2}}{(\pm 1)^m \eta^{(m+2)}(1)} \right) + o(1), \quad \text{for } a > 1,$$

$$k_n^\pm = \frac{n\pi}{a} \pm \frac{i}{2a} \log \left(\frac{-4(2n\pi i)^{m+2}}{(\pm 1)^m \eta^{(m+2)}(1)} \right) + o(1), \quad \text{for } a < 1.$$

(ii) $a = 1$ and $\int_0^1 q(x)dx \neq 0$

$$k_n^\pm = n\pi \pm \frac{i}{2} \log \left(\frac{-8(2n\pi i)^{m+1} \int_0^1 q(s)ds}{(\pm 1)^{m+1} \eta^{(m+2)}(1)} \right) + o(1).$$

Theorem 1.2. Under the assumptions in Theorem 1.1, if $a > 1$ and $\eta(r)$ is known a priori on $[\varepsilon, 1]$ with ε satisfying

$$\int_\varepsilon^1 \sqrt{\eta(r)} dr = \frac{a-1}{2}, \quad (6)$$

then $\eta(r)$ on $[0, 1]$ is uniquely determined by all zeros of $d(k)$ (including multiplicity).

Remark 1.1. Eqs.(5) and (6) lead to $\int_{\varepsilon_1}^\varepsilon \sqrt{\eta(r)} dr = 1$, which implies $\varepsilon > \varepsilon_1$.

2. PRELIMINARIES

In this section, we provide some known auxiliary results.

Using the Liouville transformation,

$$x = \int_0^r \sqrt{\eta(\rho)} d\rho, \quad \varphi(x) := (\eta(r))^{\frac{1}{4}} y(r), \quad r = r(x), \quad (7)$$

we can write the equation $y'' + k^2 \eta(r) y = 0$ with $y(0, k) = 0$ and $y'(0, k) = 1$ as

$$\varphi''(x) + (k^2 - q(x)) \varphi(x) = 0, \quad \varphi(0) = 0, \quad \varphi'(0) = \eta(0)^{-\frac{1}{4}}, \quad (8)$$

where

$$q(x) = \frac{\eta''(r)}{4(\eta(r))^2} - \frac{5}{16} \frac{(\eta'(r))^2}{(\eta(r))^3}. \quad (9)$$

Using the transformation operator theory (see, e.g. [12]), we have

$$\eta(0)^{\frac{1}{4}} \varphi(x, k) = \frac{\sin(kx)}{k} + \int_0^x K(x, t) \frac{\sin(kt)}{k} dt, \quad (10)$$

where $K(x, t)$ satisfies the following integral equation (see, e.g. [2])

$$2K(x, t) = \int_{\frac{x-t}{2}}^{\frac{x+t}{2}} q(\tau) d\tau + \int_{x-t}^x q(\tau) d\tau \int_{\tau+t-x}^{\tau} K(\tau, s) ds \\ + \int_{\frac{x-t}{2}}^{x-t} q(\tau) d\tau \int_{x-t-\tau}^{\tau} K(\tau, s) ds - \int_{\frac{x+t}{2}}^x q(\tau) d\tau \int_{x+t-\tau}^{\tau} K(\tau, s) ds, \quad (11)$$

where $0 \leq t \leq x \leq a$. In particular, $2K(x, x) = \int_0^x q(s) ds$ and $K(x, 0) = 0$. On the other hand, from Eq.(1.2.9) in [12], we know that

$$K(x, t) = K_0(x, t) - K_0(x, -t), \quad (12)$$

where $K_0(x, t)$ with $0 \leq |t| \leq x \leq a$ satisfies that if $q \in C^m[0, a]$ then $K_0(x, \cdot) \in C^{m+1}[-x, x]$ for each fixed $x \in [0, a]$ (see Theorem 1.2.2 in [12]). It follows from (12) that if $q(x)$ is smooth enough then

$$\left. \frac{\partial^{2n} K(x, t)}{\partial t^{2n}} \right|_{t=0} = 0, \quad n \in \mathbb{N}_0. \quad (13)$$

By virtue of (7) and $\eta(1) = 1$ and $\eta'(1) = 0$, we have $\varphi(a, k) = y(1, k)$ and $\varphi'(a, k) = y'(1, k)$. Thus,

$$y(1, k) = \frac{1}{\eta(0)^{\frac{1}{4}}} \left[\frac{\sin(ka)}{k} - \frac{\cos(ka)}{2k^2} \int_0^a q(s) ds + \int_0^a K_t(a, t) \frac{\cos(kt)}{k^2} dt \right], \quad (14)$$

and

$$y'(1, k) = \frac{1}{\eta(0)^{\frac{1}{4}}} \left[\cos(ka) + \frac{\sin(ka)}{2k} \int_0^a q(s) ds + \int_0^a K_x(a, t) \frac{\sin(kt)}{k} dt \right]. \quad (15)$$

Denote $K_1(t) := K_x(a, t)$ and $K_2(t) := K_t(a, t)$. Using Eq.(11), by tedious calculation, we have

$$K_1(t) = \frac{1}{4} \left[q\left(\frac{a+t}{2}\right) - q\left(\frac{a-t}{2}\right) \right] + \frac{1}{2} \int_{a-t}^a q(\tau) K(\tau, \tau+t-a) d\tau \\ - \frac{1}{2} \int_{\frac{a-t}{2}}^{a-t} q(\tau) K(\tau, a-t-\tau) d\tau + \frac{1}{2} \int_{\frac{a+t}{2}}^a q(\tau) K(\tau, a+t-\tau) d\tau, \quad (16)$$

and

$$K_2(t) = \frac{1}{4} \left[q\left(\frac{a+t}{2}\right) + q\left(\frac{a-t}{2}\right) \right] - \frac{1}{2} \int_{a-t}^a q(\tau) K(\tau, \tau+t-a) d\tau \\ + \frac{1}{2} \int_{\frac{a-t}{2}}^{a-t} q(\tau) K(\tau, a-t-\tau) d\tau + \frac{1}{2} \int_{\frac{a+t}{2}}^a q(\tau) K(\tau, a+t-\tau) d\tau. \quad (17)$$

To get Theorem 1.1, we introduce the following transcendental equation

$$z - \lambda \log z = w, \quad (18)$$

where λ is a constant. It is known (see, e.g. [8]) that Eq.(18) has a unique solution

$$z(w) = w + \lambda \log w + O\left(\frac{\log w}{w}\right) \quad (19)$$

for sufficiently large w . We will transform the equation $d(k) = 0$ to the equation with the form of (18), and then use (19) to obtain the asymptotics of non-real transmission eigenvalues. We also mention that this method, which can be used to obtain the asymptotics of non-real eigenvalues, was applied by some authors [14, 17].

For the inverse spectral problem, we shall use the following two lemmas.

Lemma 2.1. (See [9, p.28]) *Let $G(k)$ be analytic in \mathbb{C}_+ and continuous in $\overline{\mathbb{C}}_+ := \mathbb{C}_+ \cup \mathbb{R}$. Suppose that*

- (i) $\log |G(k)| = O(k)$ for $|k| \rightarrow \infty$ in $\mathbb{C}_+ := \{k \in \mathbb{C} : \text{Im} k > 0\}$,
- (ii) $|G(x)| \leq C$ for some constant $C > 0$, $x \in \mathbb{R}$,
- (iii) $\lim_{\tau \rightarrow +\infty} \log |G(i\tau)|/\tau = A$.

Then, for $k \in \overline{\mathbb{C}}_+$, there holds

$$|G(k)| \leq C e^{A \text{Im} k}.$$

Lemma 2.2. (See [13]) *For an arbitrary $0 < b < \infty$ and $p(\cdot) \in L^2[0, b]$, if $\int_0^b p(x) \varphi(x, k) \tilde{\varphi}(x, k) dx = 0$ for all $k > 0$, then $p(x) = 0$ on the interval $[0, b]$, where $\varphi(x, k)$ and $\tilde{\varphi}(x, k)$ are defined by (8) corresponding to q and \tilde{q} , respectively.*

3. PROOFS

Proof of Theorem 1.1. Rewrite Eqs.(14) and (15) as

$$y(1, k) = \frac{\sin(ka)}{\eta(0)^{\frac{1}{4}} k} [1 + P_1(k)], \quad y'(1, k) = \frac{\cos(ka)}{\eta(0)^{\frac{1}{4}}} [1 + P_2(k)], \quad (20)$$

where

$$P_1(k) = -\frac{\cot(ka)}{2k} \int_0^a q(s) ds + \frac{1}{k \sin(ka)} \int_0^a K_1(t) \cos(kt) dt, \quad (21)$$

and

$$P_2(k) = \frac{\tan(ka)}{2k} \int_0^a q(s) ds + \frac{1}{k \cos(ka)} \int_0^a K_2(t) \sin(kt) dt. \quad (22)$$

By (3), we have

$$\begin{aligned} \eta(0)^{\frac{1}{4}} d(k) &= \frac{\sin k}{k} \cos(ka) [1 + P_2(k)] - \cos k \frac{\sin(ka)}{k} [1 + P_1(k)] \\ &= \frac{\sin(k(1-a))}{2k} [2 + P_2(k) + P_1(k)] + \frac{\sin(k(1+a))}{2k} [P_2(k) - P_1(k)]. \end{aligned} \quad (23)$$

Now we shall estimate $P_2(k) - P_1(k)$ when $|k| \rightarrow \infty$ in \mathbb{C} . Since $\eta \in W_1^{m+3}[0, 1]$ with $\eta^{(u)}(1) = 0$ for $u = \overline{1, m+1}$ and $\eta^{(m+2)}(1) \neq 0$, it follows from (9) that $q \in W_1^m[0, a]$ with $q^{(u)}(a) = 0$ for $u = \overline{0, m-1}$ and $q^{(m)}(a) = \frac{\eta^{(m+2)}(1)}{4} \neq 0$. Integrating by parts in (21) and (22) for $m+1$ times, and using (13), we have

$$\begin{aligned} \int_0^a K_1(t) \cos(kt) dt &= \sin(ka) \sum_{u=0}^s \frac{K_1^{(2u)}(a)}{(-1)^u k^{2u+1}} + \cos(ka) \sum_{v=0}^{s-1} \frac{K_1^{(2v+1)}(a)}{(-1)^v k^{2v+2}} \\ &\quad + o\left(\frac{e^{|\operatorname{Im} k|a}}{k^{2s+1}}\right), \quad \text{if } m = 2s, \quad s \in \mathbb{N}, \end{aligned} \quad (24a)$$

or

$$\begin{aligned} \int_0^a K_1(t) \cos(kt) dt &= \sin(ka) \sum_{u=0}^s \frac{K_1^{(2u)}(a)}{(-1)^u k^{2u+1}} + \cos(ka) \sum_{v=0}^s \frac{K_1^{(2v+1)}(a)}{(-1)^v k^{2v+2}} \\ &\quad + o\left(\frac{e^{|\operatorname{Im} k|a}}{k^{2s+2}}\right), \quad \text{if } m = 2s+1, \quad s \in \mathbb{N}, \end{aligned} \quad (24b)$$

and

$$\begin{aligned} \int_0^a K_2(t) \sin(kt) dt &= \cos(ka) \sum_{u=0}^s \frac{K_2^{(2u)}(a)}{(-1)^{u+1} k^{2u+1}} + \sin(ka) \sum_{v=0}^{s-1} \frac{K_2^{(2v+1)}(a)}{(-1)^v k^{2v+2}} \\ &\quad + o\left(\frac{e^{|\operatorname{Im} k|a}}{k^{2s+1}}\right), \quad \text{if } m = 2s, \quad s \in \mathbb{N}, \end{aligned} \quad (25a)$$

or

$$\begin{aligned} \int_0^a K_2(t) \sin(kt) dt &= \cos(ka) \sum_{u=0}^s \frac{K_2^{(2u)}(a)}{(-1)^{u+1} k^{2u+1}} + \sin(ka) \sum_{v=0}^s \frac{K_2^{(2v+1)}(a)}{(-1)^v k^{2v+2}} \\ &\quad + o\left(\frac{e^{|\operatorname{Im} k|a}}{k^{2s+2}}\right), \quad \text{if } m = 2s+1, \quad s \in \mathbb{N}. \end{aligned} \quad (25b)$$

We only discuss the case $m = 2s$, and the case $m = 2s+1$ is similar. Substituting (24) and (25) into (21) and (22), respectively, and subtracting, we obtain

$$\begin{aligned} P_2(k) - P_1(k) &= \frac{\int_0^a q(s) ds}{2k} [\tan(ka) + \cot(ka)] + \sum_{u=0}^s \frac{K_2^{2u}(a) + K_1^{2u}(a)}{(-1)^{u+1} k^{2u+2}} \\ &\quad + \tan(ka) \sum_{v=0}^{s-1} \frac{K_2^{(2v+1)}(a)}{(-1)^v k^{2v+3}} - \cot(ka) \sum_{v=0}^{s-1} \frac{K_1^{(2v+1)}(a)}{(-1)^v k^{2v+3}} \\ &\quad + o\left(\frac{1}{k^{2s+2}}\right), \quad |k| \rightarrow \infty, \quad k \in \mathbb{C}. \end{aligned} \quad (26)$$

Denote $\mathbb{C}_+ := \{k \in \mathbb{C} : \text{Im}k > 0\}$ and $\mathbb{C}_- := \{k \in \mathbb{C} : \text{Im}k < 0\}$. Note that for $|k| \rightarrow \infty$ in \mathbb{C}_\pm ,

$$\tan(ka) = \pm i + O(e^{-2a|\text{Im}k|}), \quad \cot(ka) = \mp i + O(e^{-2a|\text{Im}k|}). \quad (27)$$

Substituting (27) into (26), and observing that $\tan(ka) + \cot(ka) = 2/\sin(2ka)$, we get

$$\begin{aligned} P_2(k) - P_1(k) &= \frac{\int_0^a q(s)ds}{k \sin(2ak)} + \sum_{u=0}^s \frac{K_2^{2u}(a) + K_1^{2u}(a)}{(-1)^{u+1} k^{2u+2}} \\ &\quad \pm i \sum_{v=0}^{s-1} \frac{K_2^{(2v+1)}(a) + K_1^{(2v+1)}(a)}{(-1)^v k^{2v+3}} + O\left(\frac{e^{-2a|\text{Im}k|}}{k^3}\right) \\ &\quad + o\left(\frac{1}{k^{2s+2}}\right), \quad |k| \rightarrow \infty, \quad k \in \mathbb{C}_\pm. \end{aligned} \quad (28)$$

Now we shall calculate $K_1^{(u)}(a) + K_2^{(u)}(a)$ for $u = \overline{0, m}$. Using (16) and (17), we have

$$K(t) := K_1(t) + K_2(t) = \frac{1}{2}q\left(\frac{a+t}{2}\right) + \int_{\frac{a+t}{2}}^a q(\tau)K(\tau, a+t-\tau)d\tau.$$

Since $q^{(u)}(a) = 0$ for $u = \overline{0, m-1}$ and $q^{(m)}(a) = \frac{\eta^{(m+2)}(1)}{4} \neq 0$, we obtain

$$K^{(u)}(a) = 0, \quad u = \overline{0, m-1}, \quad K^{(m)}(a) = \frac{q^{(m)}(a)}{2^{m+1}} = \frac{\eta^{(m+2)}(1)}{2^{m+3}}. \quad (29)$$

Substituting (29) into (28), we get, for the case $m = 2s$,

$$\begin{aligned} P_2(k) - P_1(k) &= \frac{\int_0^a q(s)ds}{k \sin(2ak)} + \frac{(-1)^{\frac{m}{2}+1} \eta^{(m+2)}(1)}{2^{m+3} k^{m+2}} \\ &\quad + O\left(\frac{e^{-2a|\text{Im}k|}}{k^3}\right) + o\left(\frac{1}{k^{m+2}}\right), \quad |k| \rightarrow \infty, \quad k \in \mathbb{C}_\pm. \end{aligned} \quad (30a)$$

Similarly, one can get that for the case $m = 2s + 1$,

$$\begin{aligned} P_2(k) - P_1(k) &= \frac{\int_0^a q(s)ds}{k \sin(2ak)} \pm i \frac{(-1)^{\frac{m-1}{2}} \eta^{(m+2)}(1)}{2^{m+3} k^{m+2}} \\ &\quad + O\left(\frac{e^{-2a|\text{Im}k|}}{k^3}\right) + o\left(\frac{1}{k^{m+2}}\right), \quad |k| \rightarrow \infty, \quad k \in \mathbb{C}_\pm. \end{aligned} \quad (30b)$$

Substituting (30) into (23), and noting that

$$\sin(k(1+a))O(e^{-2a|\text{Im}k|}) = O(e^{(1-a)\text{Im}k|}), \quad |k| \rightarrow \infty, \quad k \in \mathbb{C}_\pm,$$

we have that if $a \neq 1$ and $|k| \rightarrow \infty$ in \mathbb{C}_\pm , then, for the case $m = 2s$,

$$\eta(0)^{\frac{1}{4}}d(k) = \frac{\sin(k(1-a))}{k} \left[1 + O\left(\frac{1}{k}\right) \right] + \frac{\eta^{(m+2)}(1) \sin(k(1+a))}{(-1)^{\frac{m}{2}+1} 2(2k)^{m+3}} [1 + o(1)], \quad (31a)$$

and for the case $m = 2s + 1$,

$$\eta(0)^{\frac{1}{4}}d(k) = \frac{\sin(k(1-a))}{k} \left[1 + O\left(\frac{1}{k}\right) \right] \pm i \frac{\eta^{(m+2)}(1) \sin(k(1+a))}{(-1)^{\frac{m-1}{2}} 2(2k)^{m+3}} [1 + o(1)]. \quad (31b)$$

The remaining proof should be divided into six subcases: (i) $a > 1$ and $m = 2s$; (ii) $a > 1$ and $m = 2s + 1$; (iii) $a < 1$ and $m = 2s$; (iv) $a < 1$ and $m = 2s + 1$; (v) $a = 1$ and $m = 2s$; (vi) $a = 1$ and $m = 2s + 1$. We only discuss the subcases (i) and (v) in details, and the other cases are similar and omitted.

Case (i): by virtue of (31a), we know that $d(k) = 0$ for $|k| \rightarrow \infty$ in \mathbb{C}_\pm is equivalent to that

$$2^{m+4} k^{m+2} \sin(k(1-a)) \left[1 + O\left(\frac{1}{k}\right) \right] = (-1)^{\frac{m}{2}} \eta^{(m+2)}(1) \sin(k(1+a)) [1 + o(1)].$$

Setting $k = \frac{z}{i}$, we have $(-1)^{\frac{m}{2}} (\frac{1}{i})^{m+2} = (-1)^{\frac{m}{2}} (-1)^{\frac{m}{2}+1} = -1$, and furthermore,

$$\frac{2^{m+4} z^{m+2}}{\eta^{(m+2)}(1)} [e^{z(a-1)} - e^{z(1-a)}] = [e^{z(1+a)} - e^{-z(1+a)}] [1 + o(1)], \quad |z| \rightarrow \infty, \quad \operatorname{Re} z \neq 0,$$

Taking logarithm on both sides of the above equation, we get that for sufficiently large $n \in \mathbb{Z}$,

$$\begin{cases} z - \frac{m+2}{2} \log z = w_n, & w_n := n\pi i + \frac{1}{2} \log \left(\frac{2^{m+4}}{\eta^{(m+2)}(1)} \right) + o(1), \quad \operatorname{Re} z > 0, \\ z + \frac{m+2}{2} \log z = w_n, & w_n := n\pi i - \frac{1}{2} \log \left(\frac{2^{m+4}}{\eta^{(m+2)}(1)} \right) + o(1), \quad \operatorname{Re} z < 0. \end{cases}$$

It follows from (18) and (19) and $z = ik$ that

$$k_n^\pm = n\pi \pm \frac{i}{2} \log \left(\frac{4(2n\pi i)^{m+2}}{\eta^{(m+2)}(1)} \right) + o(1), \quad n \rightarrow \infty.$$

Case (v): if $a = 1$ and $\int_0^1 q(s)ds \neq 0$, then for the case $m = 2s$,

$$\eta(0)^{\frac{1}{4}}d(k) = \frac{\int_0^1 q(s)ds}{2k^2} \left[1 + O\left(\frac{1}{k^2}\right) \right] + \frac{\eta^{(m+2)}(1) \sin(2k)}{(-1)^{\frac{m}{2}+1} 2(2k)^{m+3}} [1 + o(1)],$$

which implies that $d(k) = 0$ for $|k| \rightarrow \infty$ in \mathbb{C}_\pm is equivalent to that

$$\frac{\int_0^1 q(s)ds}{\eta^{(m+2)}(1)} 2^{m+3} k^{m+1} (-1)^{\frac{m}{2}} = \sin(2k) [1 + o(1)], \quad |k| \rightarrow \infty, \quad k \in \mathbb{C}_\pm.$$

Setting $k = \frac{z}{i}$, we have $(-1)^{\frac{m}{2}}(\frac{1}{i})^{m+1} = (-1)^{\frac{m}{2}}(-1)^{\frac{m}{2}}\frac{1}{i} = \frac{1}{i}$, and

$$\frac{\int_0^1 q(s)ds}{\eta^{(m+2)}(1)} 2^{m+4} z^{m+1} = [e^{2z} - e^{-2z}][1 + o(1)], \quad |z| \rightarrow \infty, \quad \operatorname{Re} z \neq 0,$$

which implies that for sufficiently large $n \in \mathbb{Z}$

$$\begin{cases} z - \frac{m+1}{2} \log z = n\pi i + \frac{1}{2} \log \frac{\int_0^1 q(s)ds}{\eta^{(m+2)}(1)} 2^{m+4} + o(1), & \operatorname{Re} z > 0, \\ z + \frac{m+1}{2} \log z = n\pi i - \frac{1}{2} \log \frac{-\int_0^1 q(s)ds}{\eta^{(m+2)}(1)} 2^{m+4} + o(1), & \operatorname{Re} z < 0. \end{cases}$$

It follows from (18) and (19) and $z = ik$ that for $n \rightarrow \infty$,

$$\begin{cases} k_n^- = n\pi - \frac{i}{2} \log \left(\frac{\int_0^1 q(s)ds}{\eta^{(m+2)}(1)} 2^{m+4} (n\pi i)^{m+1} \right) + o(1), \\ k_n^+ = n\pi + \frac{i}{2} \log \left(\frac{-\int_0^1 q(s)ds}{\eta^{(m+2)}(1)} 2^{m+4} (n\pi i)^{m+1} \right) + o(1). \end{cases}$$

Through similar arguments, one obtains asymptotics of other cases. The proof is finished. \square

Proof of Theorem 1.2. Since the function $d(k)$ is an entire function of k of order 1 and even with respect to k , by Hadamard's factorization theorem,

$$d(k) = \gamma E(k), \quad E(k) := k^{2s} \prod_{k_n \neq 0} \left(1 - \frac{k^2}{k_n^2} \right), \quad (32)$$

where s is the multiplicity of the zero eigenvalue.

Using (2), (7) and (9), it can be verified that $\eta(r)$ is known a priori on $[\varepsilon, 1]$ with ε satisfying (6) is equivalent to that $q(x)$ is known a priori for $x \in [\frac{a+1}{2}, a]$. Let us prove that $q(x)$ on $[0, a]$ is uniquely determined by $E(k)$ and the known $q(x)$ on $[\frac{a+1}{2}, a]$. If it is true, then $\eta(r)$ on $[0, 1]$ with $\eta(1) = 1$ and $\eta'(1) = 0$ is uniquely determined by $E(k)$ and the known $\eta(r)$ on $[\varepsilon, 1]$. (See [11]).

Suppose that there are two functions q and \tilde{q} corresponding to the same $E(k)$ defined by (32). Let (a, φ) and $(\tilde{a}, \tilde{\varphi})$ be their corresponding quantities in (2) and (8). By virtue of (4) and $a > 1$, we obtain

$$a = \tilde{a}.$$

Denote

$$g(k) := \int_0^{\frac{a+1}{2}} [\tilde{q}(x) - q(x)] \varphi(x, k) \tilde{\varphi}(x, k) dx.$$

It follows from (10) that

$$|g(k)| \leq M_0 \frac{e^{(1+a)|\operatorname{Im} k|}}{|k|^2} \quad \text{for some } M_0 > 0. \quad (33)$$

Since $q(x) = \tilde{q}(x)$ on $[\frac{a+1}{2}, a]$, together with (8), we get

$$g(k) = \int_0^a [\tilde{q}(x) - q(x)] \varphi(x, k) \tilde{\varphi}(x, k) dx = \tilde{\varphi}'(a, k) \varphi(a, k) - \tilde{\varphi}(a, k) \varphi'(a, k). \quad (34)$$

Note that Eq.(7) with $\eta(1) = 1$ and $\eta'(1) = 0$ implies that

$$\varphi(a, k) = y(1, k) \quad \text{and} \quad \varphi'(a, k) = y'(1, k). \quad (35)$$

It yields from (3) that

$$d(k) = \frac{\sin k}{k} \varphi'(a, k) - \varphi(a, k) \cos k = \frac{\sin k}{k} [\varphi'(a, k) - \varphi(a, k) k \cot k],$$

which implies

$$\varphi'(a, k) = \frac{k}{\sin k} d(k) + \varphi(a, k) k \cot k. \quad (36)$$

Together with (36) it follows from (34) that

$$\begin{aligned} g(k) &= \frac{k}{\sin k} [\varphi(a, k) \tilde{d}(k) - \tilde{\varphi}(a, k) d(k)] \\ &= \frac{kE(k)}{\sin k} \gamma \tilde{\gamma} \left[\frac{\varphi(a, k)}{\gamma} - \frac{\tilde{\varphi}(a, k)}{\tilde{\gamma}} \right]. \end{aligned}$$

Set

$$G(k) := \frac{g(k)}{E(k)} = \frac{k}{\sin k} \gamma \tilde{\gamma} \left[\frac{\varphi(a, k)}{\gamma} - \frac{\tilde{\varphi}(a, k)}{\tilde{\gamma}} \right]. \quad (37)$$

Observing that $d(k)/\gamma = \tilde{d}(k)/\tilde{\gamma}$, one has

$$\frac{1}{\gamma} \left[\frac{\sin k}{k} \varphi'(a, k) - \varphi(a, k) \cos k \right] = \frac{1}{\tilde{\gamma}} \left[\frac{\sin k}{k} \tilde{\varphi}'(a, k) - \tilde{\varphi}(a, k) \cos k \right],$$

which implies

$$\frac{\varphi(a, n\pi)}{\gamma} - \frac{\tilde{\varphi}(a, n\pi)}{\tilde{\gamma}} = 0, \quad n = \pm 1, \pm 2, \dots,$$

and so $G(k)$ is an entire function of k from (37).

Due to (33), we know that $G(k)$ satisfies the condition (i) in Lemma 2.1. From (31) and (32) it follows that

$$E(\pm i\tau) = \frac{ce^{(a+1)\tau}}{\tau^{m+3}} [1 + o(1)], \quad c \neq 0, \quad \tau \rightarrow +\infty, \quad (38)$$

which implies from (33) and (37) that

$$|G(i\tau)| \leq C\tau^{m+1}, \quad \tau \rightarrow +\infty,$$

where $m \geq 0$ appears in Theorem 1.1. It yields $\overline{\lim}_{\tau \rightarrow +\infty} \log |G(i\tau)|/\tau := A \leq 0$.

If we can prove $|G(k)| \leq C$ for $k \in \mathbb{R}$ (see (*) below), then it follows from Lemma 2.1 that for all $k \in \overline{\mathbb{C}}_+$

$$|G(k)| \leq C. \quad (39)$$

Note that $G(k)$ is even, so Eq.(39) holds on the whole complex plane. This implies that $G(k)$ is a constant from Liouville's theorem. In addition, for the sequence $\{n\pi\}_{n \geq 1}$ there holds $G(n\pi) \rightarrow 0$ as $n \rightarrow \infty$ (see (*) below). It follows that which deduces $G(k) \equiv 0$, which implies $g(k) \equiv 0$, and so $q(x) = \tilde{q}(x)$ for $x \in [0, a]$ by Lemma 2.2.

Now, we shall prove (*): $G(k)$ is bounded on \mathbb{R} and $G(n\pi)$ tends to zero as $n \rightarrow \infty$. Using (21), (22), (23) and (32), we get

$$E(k) = \frac{\sin(k(1-a))}{k\gamma\eta(0)^{1/4}} \left[1 + O\left(\frac{1}{k}\right) \right], \quad |k| \rightarrow \infty, \quad k \in \mathbb{R},$$

which implies $\gamma\eta(0)^{1/4}$ is uniquely determine by $E(k)$ if $a \neq 1$. Substituting (10) into (37), we have

$$G(k) = \frac{\tilde{\gamma}}{\eta(0)^{1/4} \sin k} \int_0^a \left(K(a, t) - \tilde{K}(a, t) \right) \sin(kt) dt. \quad (40)$$

Note that $G(k)$ is an entire function of k from the above argument, thus, zeros of $\sin k$ can not be poles of $G(k)$. Thus, it follows from (40) that

$$\int_0^a \left(K(a, t) - \tilde{K}(a, t) \right) \sin(n\pi t) dt = 0, \quad n = 0, \pm 1, \pm 2 \dots$$

Letting $k \rightarrow n\pi$ in (40), we get from the L'Hospital principle that

$$G(n\pi) = \frac{\tilde{\gamma} \int_0^a \left(\tilde{K}(a, t) - K(a, t) \right) t \cos(n\pi t) dt}{\eta(0)^{1/4} (-1)^n}, \quad n = 0, \pm 1, \pm 2 \dots \quad (41)$$

Thus, $G(k)$ is bounded on \mathbb{R} and $G(n\pi)$ tends to zero as $n \rightarrow \infty$ from (41). Therefore, we have finished the proof. \square

Acknowledgments. The author Xu was supported by Innovation Program for Graduate Students of Jiangsu Province of China (KYLX16_0412). The authors Xu and Yang were supported in part by the National Natural Science Foundation of China (11171152 and 91538108) and the Natural Science Foundation of Jiangsu Province of China (BK 20141392). The author Buterin was supported in part by RFBR (Grants 15-01-04864). The authors Buterin and Yurko were supported by the Ministry of Education and Science of RF (Grant 1.1660.2017/PCh) and by RFBR (16-01-00015 and 17-51-53180).

REFERENCES

- [1] T. Aktosun, D. Gintides, V.G. Papanicolaou, The uniqueness in the inverse problem for transmission eigenvalues for the spherically symmetric variable-speed wave equation, *Inverse Problems* **27** (2011), 115004 (17pp).
- [2] S.A. Buterin, C.-F. Yang, V.A. Yurko, On an open question in the inverse transmission eigenvalue problem, *Inverse Problems* **31** (2015), 045003 (8pp).
- [3] S.A. Buterin, C.-F. Yang, On an inverse transmission problem from complex eigenvalues, *Results. Math.* (2015). <http://dx.doi.org/10.1007/s00025-015-0512-9>

- [4] L.-H. Chen, On the inverse spectral theory in a non-homogeneous interior transmission problem, *Complex Variables and Elliptic Equations*, **60** (2015), 707-731.
- [5] D. Colton, Y.J. Leung, Complex eigenvalues and the inverse spectral problem for transmission eigenvalues, *Inverse Problems* **29** (2013), 104008 (6pp).
- [6] D. Colton, Y.J. Leung, S.X. Meng, Distribution of complex transmission eigenvalues for spherically stratified media, *Inverse Problems* **31** (2015), 035006 (19pp).
- [7] D. Colton, Y.J. Leung, The existence of complex transmission eigenvalues for spherically stratified media, *Applicable Analysis* **96** (2017), 39-47.
- [8] M.V. Fedoryuk, *Asymptotics: Integrals and Series*, Nauka, Moscow, 1987. (Russian)
- [9] P. Koosis, *The Logarithmic Integral I*, Cambridge University Press, Cambridge, 1988.
- [10] B.Ja. Levin, *Lectures on Entire Functions*. Translation of Mathematical Monographs. Providence (RI): AMS, 1996.
- [11] J.R. McLaughlin and P.L. Polyakov, On the uniqueness of a spherically symmetric speed of sound from transmission eigenvalues, *J. Diff. Eqns.* **107** (1994), 351-382.
- [12] V. Marchenko, *Sturm-Liouville Operators and Applications*. Publisher Birkhäuser, Boston, 1986.
- [13] A.G. Ramm, Property C for ODE and applications to inverse problems, *Fields Institute Communications*, Providence, RI, **25**, (2000), 15-75.
- [14] S.A. Stepin, A.G. Tarasov, Asymptotic distribution of resonances for one-dimensional Schrödinger operators with compactly supported potential, *Sbornik. Mathematics*, **198** (2007), 87-104.
- [15] G. Wei, H.-K. Xu, Inverse spectral analysis for the transmission eigenvalue problem, *Inverse Problems* **29** (2013), 115012 (24pp).
- [16] X.-C. Xu, X.-X. Xu, C.-F. Yang, Distribution of transmission eigenvalues and inverse spectral analysis with partial information on the refractive index, *Math. Meth. Appl. Sci.* **39** (2016), 5330-5342.
- [17] X.-C. Xu, C.-F. Yang, H.-Z. You, Inverse spectral analysis for Regge problem with partial information on the potential, *Results. Math.* (2016). <http://dx.doi.org/10.1007/s00025-015-0523-6>
- [18] C.-F. Yang, S.A. Buterin, Uniqueness of the interior transmission problem with partial information on the potential and eigenvalues, *J. Diff. Eqns.* **260** (2016), 4871-4887.